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Towards geometry for realistic stars from recursions

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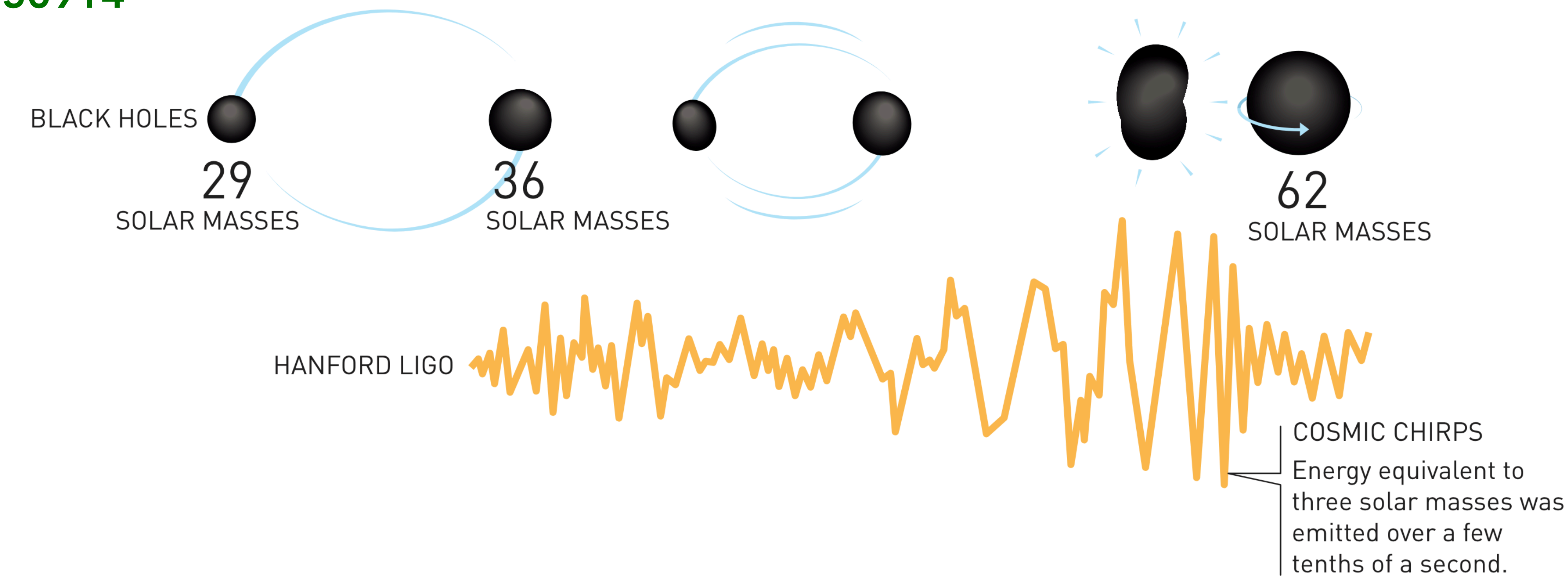
Gravitational wave from Binary BH mergers

GW150914



- **Gravitational wave: new window to probe our Universe**
- How do we describe this system? $\Longrightarrow$ **Solve Einstein equation** (perturbatively)
- ***Are the theoretical tools we have powerful enough to solve this problem?***

Toy model — Schwarzschild BH solution

Solving perturbative Einstein Equation

1. **Solve Einstein Equation directly** (old and brute force approach)
   - Green function method
   - Perturbative GR is notorious for its complexity
   - Leading order correction is the practical limit [Florides, Synge 61] [Westpfahl, 85]
2. **Scattering amplitude Approach** (since 2018)
   - Modern techniques in **QFT/Quantum Gravity**

     Generalized unitarity Bern, Dixon, Dunbar, Kosower, hep-ph/9403226

     On-shell recursion Britto, Cachazo, Feng, Witten, hep-th/0501052

     Color-kinematics duality and double copy Bern, Carrasco, Johansson, 0805.3993, 1004.0476
   - Issues — convergence of the series, loop integrals, etc
3. **Go back to the Einstein equation again** (armed with new techniques)
   - Most efficient method (biased) for solving perturbative GR [Damgaard, KL `24]

Traditional approach

Solving EoM perturbatively

- ϕ^4**-theory case**: $\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$

$$(-\Box + m^2)\phi(x) = \lambda\phi(x)^3, \qquad \lambda \ll 1$$

- We may expand the solution as a power series in λ: $\phi = \sum_{n=0}^{\infty} \phi_n \lambda^n$

- EoM at λ^n-order

$$\left(-\Box + m^2\right)\phi_n = \sum_{\substack{k,l,m \\ k+l+m+1=n}} \phi_k\phi_l\phi_m$$

- We may solve the equation by using **Green's function** $G(x-y)$ iteratively

$$(-\Box + m^2)\phi_0 = 0 \longleftarrow \text{plane-wave solution}$$

$$\phi_1 = \int_y G(x-y)\left(\phi_0(y)\right)^3,$$

retarded Green's ft

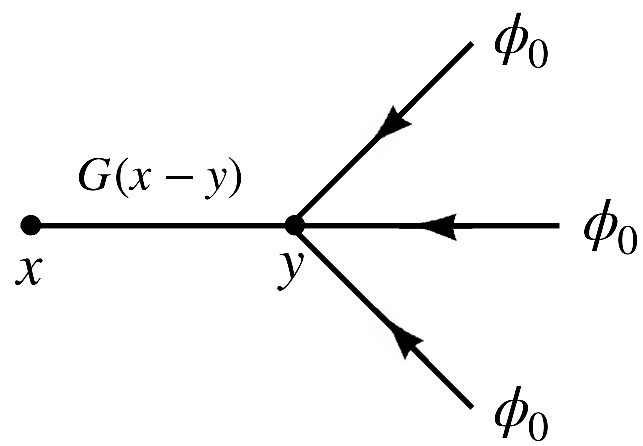
$$\phi_2 = \int_y 3G(x-y)\left(\phi_0(y)\right)^2\phi_1 = \int_y\int_{y'} 3G(x-y)\phi_0(y)^2 G(y-y')\phi_0\left(y'\right)^3,$$

Diagrammatic representation

- One may find a simple pattern $\Longrightarrow$ **Feynman diagram (tree level)**

$$\phi_1 = \int_y G(x-y)\big(\phi_0(y)\big)^3$$

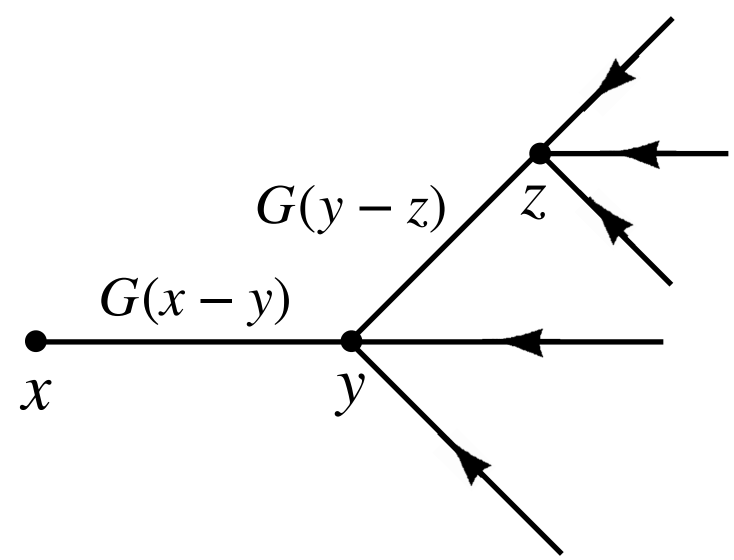
$\longleftrightarrow$

$\phi_1 =$ 

loop(?!) integrals

$$\phi_2 = \int_y 3G(x-y)\big(\phi_0(y)\big)^2\phi_1(y)$$
$$= \int_y \int_z 3G(x-y)\phi_0(y)^2 G(y-z)\phi_0(z)^3$$

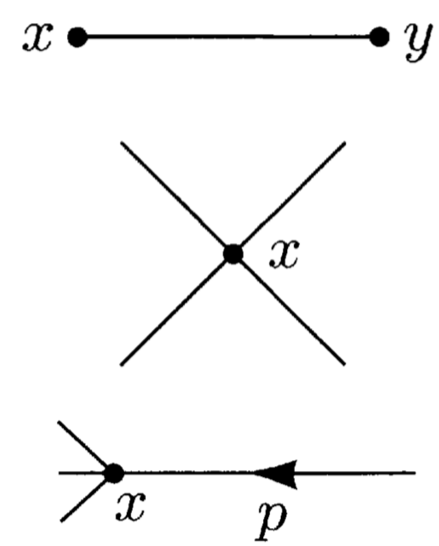
$\longleftrightarrow$

$\phi_2 = 3$ 

Diagrams = EoM solution

Feynman Rules and gravity

- Feynman Rules



Propagator $= G(x-y)$

Vertex $= (-i\lambda)\int d^4x$

External leg $= e^{ip\cdot x}$

- Diagrams generate "*loop integrands*" We don't need to derive EoM!
- Incorporating the **QFT techniques** — **Feynman diagrams, loop integration…**
- ***Generalization to the gravity* — *Feynman diagrams for GR*** (impossible task)
- **Schwarzschild BH solution with a nontrivial source, "Quantum tree graphs"** [Duff `73]

Complexity of Perturbative GR

- For **weak field regimes** $|h| \ll 1$

Flat background, no curvature

Metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ Fluctuation, graviton

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + (h^2)^{\mu\nu} - (h^3)^{\mu\nu} \cdots \qquad \sqrt{-g} = 1 - \frac{1}{2}\,\mathrm{tr}\,h + \frac{1}{4}\left(\frac{h^2}{2} - \mathrm{tr}\left[h^2\right]\right) + \cdots$$

- **Einstein-Hibert action**:

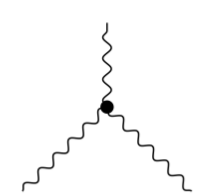
$$S_{\mathrm{EH}} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{4}g^{\mu\nu}\partial_\mu g^{\rho\sigma}\partial_\nu g_{\rho\sigma} - \frac{1}{2}g^{\mu\nu}\partial_\mu g^{\rho\sigma}\partial_\rho g_{\nu\sigma} - g^{\mu\nu}\partial_\mu\partial_\nu \ln\sqrt{-g}\right]$$

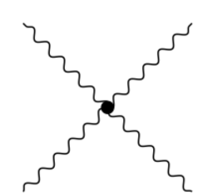
$$\mathcal{O}(h^2) = 4, \qquad \mathcal{O}(h^3) = 13, \qquad \mathcal{O}(h^4) = 35, \qquad \mathcal{O}(h^5) = 76 \ldots$$

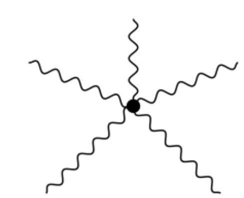
- **Feynman rules?!** Bottleneck of the quantum gravity.

# of terms	2	~ 13*3!=78	~ 35*4!=840	~ 76*5!=9120

Propagator







Amplitude approach since 2018

- **Iwasaki ('71)** noticed that the classical limit, $\hbar \to 0$, of quantum scattering amplitudes leads to the classical gravity contribution — computed the **binding potential** between two BHs by using 2 to 2 scalar amplitude (Lippmann-Schwinger equation)

- Not all diagrams are relevant to the classical results at one-loop

box (II)   crossed box ($\overline{\mathrm{II}}$)   triangle ($\triangle$)   inverted triangle ($\bigtriangledown$)

- **Loophole: careful with the classical limit $\hbar \to 0$ of massive field theories**

- See the Klein-Gordon equation for a massive scalar field: $\Box \phi(x) + \frac{m^2}{\hbar^2}\phi(x) = 0$

- **Loop order $\neq$ $\hbar$ order**, Quantum gravity computation (loop diagrams).

- Recently, this approach has been revived by the **development of the amplitude techniques.** [Neill, Rothstein 13, Bjerrum-Bohr, Damgaard, Festuccia, Plante, Vanhove, Cheung, Rothstein, Solon, Kosower, Maybee, O'Connell `18,…]

In this talk…

- Returning to solving Einstein equation explicitly **[Damgaard, KL, PRL `24]**

PRL 2024 COLLECTION

- **Three main ideas**
  - **good variable** — By doubling the fields, the perturbative Einstein equation is drastically simplified. We can hide the ugly infinite expansion.
  - **recursion** — A new methodology for solving perturbative Einstein Equation
  - **bubble integrals** — All the "higher-loop integrals" are represented by iterations of one-loop integrals

- For the Schwarzschild BH case, we derived **all-order results** — first derivation!
  - **Efficiency** — fixed number of terms, recursions and simple loop integrals…
  - **Universality** — binary black holes, rotating black holes & cosmology etc

Perturbative GR
and
doubling prescription

Tensor density representation

- **Two sources** of the infinite expansion: g^{-1} and $\sqrt{-g}$
- **Field redefinition - tensor density** [Landau & Lifshitz book]:

$$\sigma^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}\,, \qquad \sigma_{\mu\nu} = \frac{1}{\sqrt{-g}} g_{\mu\nu}\,,$$

- **EH action** (up to total derivative) in terms of the **tensor density**

$$S_{\rm EH} = \int \mathrm{d}^D x \left[\frac{1}{4}\sigma^{\mu\nu}\partial_\mu\sigma^{\rho\sigma}\partial_\nu\sigma_{\rho\sigma} - \frac{1}{2}\sigma^{\mu\nu}\partial_\mu\sigma^{\rho\sigma}\partial_\rho\sigma_{\nu\sigma} + (D-2)\sigma^{\mu\nu}\partial_\mu\hat{d}\partial_\nu\hat{d} \right], \qquad \partial_\mu\hat{d} = -\frac{1}{4}\sigma^{\rho\sigma}\partial_\mu\sigma_{\rho\sigma}$$

- **Idea:** do not substitute metric perturbations from the beginning!
- Let us treat the **metric** (σ) and the **inverse metric** (σ^{-1}) on **equal footing**
- **Remove metric** (σ) and **introduce an auxiliary field $\tilde{\sigma}$. on-shell value of $\tilde{\sigma}_{\mu\nu} = \sigma_{\mu\nu}$**
- Impose a **constraint:** $\tilde{\sigma}_{\mu\nu}\sigma^{\nu\rho} = \delta_\mu{}^\rho$
- perturbative expansions: $\sigma^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}\,, \quad \tilde{\sigma}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}\,.$
- Then $\tilde{h}$ satisfies the constraint $\Longrightarrow \tilde{h}^{\mu\nu} = h^{\mu\nu} + \tilde{h}^\mu{}_\rho h^{\rho\nu}\,, \qquad \tilde{h}^{\mu\nu}_{(n)} = h^{\mu\nu}_{(n)} + \sum_{m=1}^{n-1} \tilde{h}^{\mu\rho}_{(n-m)} h^{\rho\nu}_{(m)}$

Source of the Schwarzschild BH

- Consider pure gravity with a matter

$$S=\int \mathrm{d}^4x\left[\frac{1}{2\kappa^2}\sqrt{-g}R+\frac{1}{2}j_{\mu\nu}(x)g^{\mu\nu}(x)\right]$$

where $j_{\mu\nu}(x)$ is an external source (density) without metric dependence,

- Relation to the energy-momentum tensor $T_{\mu\nu}$

$$\sqrt{-g}T_{\mu\nu}=j_{\mu\nu}$$

- Schwarzschild BH is not a vacuum solution — point mass source

- **Energy-momentum tensor** for a point mass traveling on a worldline $x^\mu(\tau)$

$$T^{\mu\nu}\left(y^\sigma\right)=8\pi GM\int\left[\frac{\delta^{(4)}\left(y^\sigma-x^\sigma(\tau)\right)}{\sqrt{-g}}\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau$$

- Source of Schwarzschild BH — a **static point mass** placed at the origin, $\boldsymbol{x}=0$

$$j_{\mu\nu}(x)=8\pi GMv_\mu v_\nu\delta^3(\boldsymbol{x})\,,\qquad v^\mu=\frac{dx^\mu}{d\tau}=(-1,0,0,0)\,.$$

Field equation

- **Einstein tensor (density)**

$$\begin{aligned}\mathscr{G}^{\mu\nu} = &\ \frac{1}{2}\sigma^{\rho\sigma}\left[\partial_\rho\partial_\sigma\sigma^{\mu\nu} + \partial_\rho\sigma^{\kappa\mu}\partial_\sigma\tilde{\sigma}_{\kappa\lambda}\sigma^{\nu\lambda}\right] - \sigma^{\rho(\mu}\left[\partial_\rho\partial_\sigma\sigma^{\nu)\sigma} + \partial_\rho\sigma^{|\kappa\lambda}\partial_\kappa\tilde{\sigma}_{\lambda\sigma}\sigma^{\sigma|\nu)}\right] \\ &+\sigma^{\mu\kappa}\sigma^{\nu\lambda}\left[\frac{1}{4}\partial_\kappa\sigma^{\rho\sigma}\partial_\lambda\tilde{\sigma}_{\rho\sigma} + (D-2)\partial_\kappa\hat{d}\partial_\lambda\hat{d}\right] \\ &+\frac{1}{2}\left[\partial_\rho\sigma^{\rho\sigma}\partial_\sigma\sigma^{\mu\nu} - \partial_\sigma\sigma^{\rho\mu}\partial_\rho\sigma^{\sigma\nu}\right] + \sigma^{\mu\nu}\left[\partial_\kappa\left(\sigma^{\kappa\lambda}\partial_\lambda\hat{d}\right)\right] ,\end{aligned}$$

- **Einstein equation**

$$\sqrt{-\sigma}\mathscr{G}^{\mu\nu} = j^{\mu\nu}$$

$$\mathscr{G}^{\mu\nu} = \sum_{n=1}^{\infty} G^n \mathscr{G}^{\mu\nu}_{(n)} , \qquad j^{\mu\nu}(x) = G j^{\mu\nu}_{(1)}(x)$$

- $j^{\mu\nu}$ **contributes to the** G^1**-order only**

$$\mathscr{G}^{\mu\nu}_{(1)} = -\frac{1}{2}\Box h^{\mu\nu}_{(1)} = j^{\mu\nu}$$

$$\mathscr{G}^{\mu\nu}_{(n)} = 0 , \qquad n > 1$$

Harmonic vs de Donder gauge

- One of the most straightforward gauge choices is the **harmonic** or **de Donder** **gauge**

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = 0 \qquad \text{or} \qquad \partial_\mu h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu h^{\rho}{}_{\rho} = 0 \qquad \text{for } g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

harmonic gauge — de Donder gauge

- **Linearized harmonic gauge** = **de Donder gauge**, but not in higher orders
- However, in the tensor density perturbations, these are equivalent

$$g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = \partial_\mu\left(\sqrt{-g}\,g^{\mu\rho}\right) = \partial_\mu \sigma^{\mu\rho} = \partial_\mu h^{\mu\rho} = 0$$

- In our perturbation convention,

harmonic gauge = de Donder gauge

- If we obtained a solution using amplitude, in what coordinates do we get the result?

$\Longleftrightarrow$ **gauge choice**

- **However, it is not obvious in actual computation…**

Schwarzschild metric in harmonic coordinates

- The usual form of the Schwarzischild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- In the **harmonic coordinates**, the metric

$$ds^2 = -\frac{r - GM}{r + GM} dt^2 + \frac{r + GM}{r - GM} dr^2 + (r + GM)^2\, d\Omega\,, \qquad \text{obtained by } r \to r + GM$$

- The **tensor density $\sigma^{\mu\nu}$** for this metric ($\sigma^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$),

$$\sigma^{\mu\nu}\partial_\mu\partial_\nu = -\frac{(r + GM)^3}{r^2\,(r - GM)}\partial_t^2 + \left(\delta^{ij} - \frac{G^2M^2x^ix^j}{r^4}\right)\partial_i\partial_j\,.$$

- The corresponding metric perturbations $h^{\mu\nu}$

$$h^{00} = -1 + \frac{(r + GM)^3}{r^2\,(r - GM)} = \frac{4GM}{r} + \frac{7G^2M^2}{r^2} + \frac{8G^3M^3}{r^3} + \frac{8G^4M^4}{r^4} + \cdots\,,$$

$$h^{ij} = \frac{G^2M^2x^ix^j}{r^4}\,.$$

Coefficients of h^{00} is fixed by "8" while h^{ij} truncates at the second order

Remained ambiguity

[Fromholz, Poisson, Will `13
The Schwarzschild metric: It's the coordinates, stupid!]

- Even after the harmonic gauge, the form of the metric is not fixed yet
- Most general solution in harmonic coordinate — a new parameter C

$$\sigma^{00} = -1 - \frac{4M}{r} - \frac{7M^2}{r^2} - \frac{8M^3}{r^3} - \frac{8M^4 - 2CM/3}{r^4} + \mathcal{O}\left(r^{-5}\right)$$

$$\sigma^{ij} = \left(1 - \frac{C}{3r^3} - \frac{2CM^2}{5r^5} + O\left(r^{-6}\right)\right)\delta^{ij} + \left(-\frac{G^2M^2}{r^2} + \frac{C}{r^3} + \frac{2G^2M^2C}{3r^3} + \mathcal{O}\left(r^{-6}\right)\right)\frac{x^i x^j}{r^2}$$

- Solving the de Donder gauge, $\partial_\mu h^{\mu\nu} = 0$, admits an **integration constant C**
- If we turn off C, the solution returns to the previous metric expansion.
- The existence of the parameter has recently been observed in the differential equation.
- *How can we interpret this ambiguity in **the field theory context**?*

Recursion Relation
for perturbative GR solutions

Perturbiner expansion [Rosly, Selivanov '96,'97], [KL '22]

- **Modern derivation:** substituting the **perturbiner expansion** into the classical **EoM**

  $\Longrightarrow$ connects **solutions of EoM** and **tree-level amplitudes**

- **The classical field** in the quantum effective action formalism — 1-point function in the presence of the source $j^{\mu\nu}$

$$h^{\mu\nu}(\boldsymbol{x}) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{y_1, y_2, \cdots, y_n} \left\langle 0 \middle| T\left[h_x^{\mu\nu} h_{y_1}^{\kappa_1 \lambda_1} \cdots h_{y_n}^{\kappa_n \lambda_n} \right] \middle| 0 \right\rangle_c \frac{i j_{y_1}^{\kappa_1 \lambda_1}}{\hbar} \cdots \frac{i j_{y_n}^{\kappa_n \lambda_n}}{\hbar} .$$

- The field corresponds to a different physical quantity depending on the sources:

  - **Inverse propagator:** $j_x^{\mu\nu} = \sum_{i=1}^{N} \int_{y_i} \mathbf{K}_{xy_i}^{\mu\nu,\rho\sigma} e^{-ik_i \cdot y_i}$ $\Longrightarrow$ scattering amplitude.
  - **Plane-wave:** $j_x^{\mu\nu} = \sum_{i=1}^{N} \int_{y_i} e^{-ik_i \cdot y_i}$ $\Longrightarrow$ **Correlation function.**
  - **Point-mass source:** $j_{\boldsymbol{x}}^{\mu\nu} = M v^\mu v^\nu \int_{\ell} e^{-i\boldsymbol{\ell}\cdot\boldsymbol{x}}$ $\Longrightarrow$ **solutions of EoM**. $v^\mu = \dfrac{dx^\mu}{d\tau} = (-1,0,0,0)\,.$

Perturbiner expansion for classical solutions

- Substituting the external sources: $h^{\mu\nu}(\boldsymbol{x}) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\ell_1,\ell_2,\cdots,\ell_n} J^{\mu\nu}_{\ell_1\ell_2\cdots\ell_n} e^{-i\ell_{12\cdots n}\cdot \boldsymbol{x}}$,

- It is convenient to shift the loop momenta, $\ell_1 \to -\ell_{12\cdots n}$

$$h^{\mu\nu}(\boldsymbol{x}) = \sum_{n=1}^{\infty} \int_{\ell_1} e^{i\ell_1\cdot \boldsymbol{x}} \int_{\ell_2,\cdots,\ell_n} \frac{1}{(n-1)!} J^{\mu\nu}_{-\ell_{12\cdots n}\ell_2\cdots\ell_n} = \sum_{n=1}^{\infty} \int_{\ell_1} e^{i\ell_1\cdot \boldsymbol{x}} J^{\mu\nu}_{(n)|\ell_1},$$

$$J^{\mu\nu}_{(n)|\ell_1} = \int_{\ell_2,\cdots,\ell_n} \frac{1}{(n-1)!} J^{\mu\nu}_{-\ell_{12\cdots n}\ell_2\cdots\ell_n}$$

- Compare with the **amplitude perturbiner — A continuous limit**

  finite # of particles cannot generate the classical solutions

$$h^{\mu\nu} = \sum_{\mathscr{P}} J^{\mu\nu}_{\mathscr{P}} e^{-ik_{\mathscr{P}}\cdot x}$$

- We call the number of the loop momenta of an off-shell current as **rank**.

  Here the rank is equivalent to the powers of coupling G

$$h^{\mu\nu} = \sum_{n=0}^{\infty} G^n h^{\mu\nu}_{(n)} \qquad \text{and} \qquad h^{\mu\nu}_{(n)} = \int_{\ell} J^{\mu\nu}_{(n)|\ell} e^{i\ell\cdot \boldsymbol{x}}$$

Structure of loop integrals

- Substituting the perturbiner expansion into the EoMs

$$h^{\mu\nu}_{(n)} = \int_{\ell} e^{i\ell\cdot x} J^{\mu\nu}_{(n)|\ell} \quad \text{and} \quad \tilde{h}^{\mu\nu}_{(n)} = \int_{\ell} e^{i\ell\cdot x} \tilde{J}^{\mu\nu}_{(n)|\ell}$$

- **Perturbative Einstein eq**

$$h_1(x)h_2(x)\cdots h_n(x) \Longrightarrow \int_{\ell_1} e^{i\ell_{12\cdots n}\cdot x} \int_{\ell_2,\ell_3,\cdots\ell_n} J_{1|\ell_1}J_{2|\ell_2}\cdots J_{n|\ell_n} = \int_{\ell_1} e^{-i\ell_1\cdot x} \int_{\ell_2,\ell_3,\cdots\ell_n} J_{1|-\ell_{12\cdots n}}J_{2|\ell_2}\cdots J_{n|\ell_n}$$

$$\int_{\ell_3,\cdots\ell_n} \left(\underline{\int_{\ell_2} J_{1|-\ell_{12\cdots n}} J_{2|\ell_2}} \right) J_{3|\ell_3}\cdots J_{n|\ell_n}$$

One-loop bubble integral

$J'_{1|\ell_1\ell_3\cdots\ell_n}$

$$\int_{\ell_4,\cdots\ell_n} \left(\int_{\ell_3} J'_{1|-\ell_{13\cdots n}} J_{3|\ell_3} \right) J_{4|\ell_4}\cdots J_{n|\ell_n}$$

- **Fourier integrals $\Longleftrightarrow$ loop integrals: number of loops = number of fields - 1**
- **Integral Factorization —** iterative structure of loop integrals.
- This implies that **only bubble integrals are required**

Recursions and currents at rank 1

- Rank-1 EoM — **Poisson equation**

$$\Delta h^{\mu\nu}_{(1)} = -\, 2j^{\mu\nu} = -\, 2Mv^{\mu}v^{\nu}\int_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

- Substituting the perturbiner expansion $h^{\mu\nu}_{(1)} = \int_{\boldsymbol{\ell}} J^{\mu\nu}_{\boldsymbol{\ell}} e^{-i\boldsymbol{\ell}\cdot\boldsymbol{x}}$, we obtain the initial condition of the off-shell recursion relation

$$J^{\mu\nu}_{(1)|\boldsymbol{\ell}} = \frac{2\kappa^2 M}{|\boldsymbol{\ell}|^2} v^{\mu}v^{\nu} = \frac{16\pi GM}{|\boldsymbol{\ell}|^2} v^{\mu}v^{\nu}\,,$$

Or equivalently

$$J^{00}_{(1)|\boldsymbol{\ell}} = \frac{16\pi GM}{|\boldsymbol{\ell}|^2}\,, \qquad J^{0i}_{(1)|\boldsymbol{\ell}} = 0\,, \qquad J^{ij}_{(1)|\boldsymbol{\ell}} = 0\,.$$

- Since we are assuming an asymptotically flat metric, J^{ij} cannot be a plane wave.
- After Fourier transformation, we have the Newton potential — consistent with the metric expansion

$$h^{00}_{(1)} = \frac{4GM}{r} \qquad h^{0i}_{(1)} = 0 \qquad h^{ij}_{(1)} = 0$$

Recursions and currents at rank 2

- The corresponding recursion is

$$J^{00}_{(2)|-\ell_1} = \frac{\kappa}{|\boldsymbol{\ell}_1|^2}\int_{\ell_2}\left[\frac{5}{4}|\boldsymbol{\ell}_2|^2 - \frac{7}{8}\boldsymbol{\ell}_{12}\cdot\boldsymbol{\ell}_2\right]J^{00}_{(1)|-\ell_{12}}J^{00}_{(1)|\ell_2}\,,$$

$$J^{ij}_{(2)|-\ell_1} = \frac{\kappa}{|\boldsymbol{\ell}_1|^2}\int_{\ell_2}\left[\frac{\ell^{(i}_{12}\ell^{j)}_2}{4} - \frac{\delta^{ij}\boldsymbol{\ell}_{12}\cdot\boldsymbol{\ell}_2}{8}\right]J^{00}_{(1)|-\ell_{12}}J^{00}_{(1)|\ell_2}\,.$$

- 1-loop **bubble integrals**

$$J^{00}_{(2)|-\ell_1} = \frac{(16\pi GM)^2}{|\boldsymbol{\ell}_1|^2}\int_{\ell_2}\frac{1}{|\boldsymbol{\ell}_2|^2|\boldsymbol{\ell}_{12}|^2}\left[\frac{5}{4}|\boldsymbol{\ell}_2|^2 - \frac{7}{8}\boldsymbol{\ell}_{12}\cdot\boldsymbol{\ell}_2\right] = \frac{14\pi^2G^2M^2}{|\boldsymbol{\ell}_1|}\,.$$

$$J^{ij}_{(2)|-\ell_1} = \frac{(16\pi GM)^2}{8|\boldsymbol{\ell}_1|^2}\int_{\ell_2}\left[\frac{2\ell^{(i}_1\ell^{j)}_2 + 2\ell^i_2\ell^j_2 - \delta^{ij}\ell^k_1\ell^k_2}{|\boldsymbol{\ell}_2|^2|\boldsymbol{\ell}_{12}|^2} + \frac{\delta^{ij}}{2}\frac{1}{|\boldsymbol{\ell}_{12}|^2}\right] = \pi^2G^2M^2\left[-\frac{\ell^i_1\ell^j_1}{|\boldsymbol{\ell}_1|^3} + \frac{\delta^{ij}}{|\boldsymbol{\ell}_1|}\right]$$

- The Fourier transformation gives the correct perturbed metric

Recursions and currents at rank 3

- **Rank-3 recursion**

$$|\boldsymbol{\ell}_1|^2 J^{00}_{(3)|-\boldsymbol{\ell}_1} = -(GM)^3\left[\ell_1^i X^i_{(3)|-\boldsymbol{\ell}_1} + \ell_1^i \int_{\ell_2} \ell_{12}^j J^{00}_{(1)|-\boldsymbol{\ell}_{12}} J^{ij}_{(2)|\boldsymbol{\ell}_2}\right].$$

$$|\boldsymbol{\ell}_1|^2 J^{ij}_{(3)|-\boldsymbol{\ell}_1} = \int_{\boldsymbol{\ell}_2}\left[8d^{(i}_{(2)|-\boldsymbol{\ell}_{12}} d^{j)}_{(1)|\boldsymbol{\ell}_2} - 2h^{ij}_{(2)}\ell_2^k d^k_{(1)|\boldsymbol{\ell}_2}\right] + 2\delta^{ij}\ell_1^k d^k_{(3)} + \frac{1}{2}W^{ij}_{(3)}$$

$$X^i_{(n)|-\boldsymbol{\ell}_1} = \int_{\boldsymbol{\ell}_2} \ell_2^i \sum_{m=1}^{n-1} \tilde{J}^{00}_{(n-m)|-\boldsymbol{\ell}_{12}} J^{00}_{(m)|\boldsymbol{\ell}_2}, \qquad Y^i_{(n)|-\boldsymbol{\ell}_1} = \int_{\boldsymbol{\ell}_2} \ell_2^i \tilde{J}^{kl}_{(n-2)|-\boldsymbol{\ell}_{12}} J^{kl}_{(2)|\boldsymbol{\ell}_2},$$

- Again, we need only 1-loop bubble integrals.
- In dimensional regularization

Scaleless integral
vanishes in dim. Reg.

$$J^{00}_{(3)|-\boldsymbol{\ell}_1} = \frac{(GM)^3}{|\boldsymbol{\ell}_1|^{d-3}} 2^{d+1}\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d-3}{2}\right), \qquad J^{ij}_{(3)|\boldsymbol{\ell}} = -\int_{\boldsymbol{\ell}_2}\frac{16\pi^3\delta^{ij}}{3|\boldsymbol{\ell}_1|^2|\boldsymbol{\ell}_2|} = 0\,.$$

- The only place where divergences arise!
- **other regularization scheme** — it does not vanish, and the solution should be modified!
- This explains the ambiguity, C factor

Recursions and currents at rank 4

- Rank-4 EoMs

$$\Delta h^{00}_{(4)} = \partial_i\left(\sum_{m=1}^{3} \tilde{h}^{00}_{(4-m)}\partial^i h^{00}_{(m)} - \tilde{h}^{kl}_{(2)}\partial^i h^{kl}_{(2)} - \tilde{h}^{00}_{(1)}\partial_j h^{00}_{(1)} h^{ij}_{(2)} + \partial_j h^{00}_{(2)} h^{ij}_{(2)} + \partial_j h^{kk}_{(2)} h^{ij}_{(2)}\right),$$

$$\begin{aligned}\Delta h^{ij}_{(4)} = {}& h^{kl}_{(2)}\partial_k\partial_l h^{ij}_{(2)} - 2h^{k(i}_{(2)}\partial_k\partial_l h^{j)l}_{(2)} + \frac{1}{2}\left(2Z^{k(i}_{(4)}{}_{k}{}^{j)} - 4Z^{(i|k|}_{(4)}{}_{k}{}^{j)} + Z^{(i|k|j)}_{(4)}{}_{k} + W^{(ij)}_{(4)}\right)\\ & -\frac{1}{2}W^{(i}_{(2)l}h^{j)l}_{(2)} + 4d^{i}_{(2)}d^{j}_{(2)} + \partial_k h^{kl}_{(2)}\partial_l h^{ij}_{(2)} - \partial_l h^{ki}_{(2)}\partial_k h^{lj}_{(2)} - 2h^{ij}_{(2)}\partial_k d^{k}_{(2)}\,,\end{aligned}$$

- 3-loop integral in the diagrammatic approach — (1-loop)^3
- The solution of the corresponding recursion

$$J^{00}_{(4)|-\ell_1} = (GM)^4 2^{d-1}\pi^{d/2}\,|\,\ell_1\,|^{4-d}\,\Gamma\left(\tfrac{d}{2}-2\right),$$

$$J^{ij}_{(4)|-\ell_1} = 0$$

All-order Currents

- From $n \geq 5$ cases, the forms of the EoM/Recursion are fixed.
- In the harmonic gauge, the Landau-Lifshitz variables are extremely simple

$$h^{00} = -1 + \frac{(r+GM)^3}{r^2(r-GM)} = \frac{4GM}{r} + \frac{7G^2M^2}{r^2} + \frac{8G^3M^3}{r^3} + \frac{8G^4M^4}{r^4} + \cdots ,$$
$$h^{ij} = \frac{G^2M^2x^ix^j}{r^4} .$$

- One can read off the currents arbitrary order in G from the Fourier transformation

$$J^{00}_{(1)|\boldsymbol{\ell}} = \frac{4(GM)2^{D-1}\pi^{\frac{D}{2}}\Gamma\left[\frac{D-1}{2}\right]}{\Gamma[\frac{1}{2}]}\frac{1}{|\boldsymbol{\ell}|^{D-1}} ,$$
$$J^{00}_{(2)|\boldsymbol{\ell}} = 7(GM)^2 2^{D-2}\pi^{\frac{D}{2}}\Gamma[\tfrac{D-2}{2}]\frac{1}{|\boldsymbol{\ell}|^{D-2}} ,$$
$$J^{00}_{(n)|\boldsymbol{\ell}} = \frac{8(GM)^n\pi^{\frac{D}{2}}\Gamma\left[\frac{D-n}{2}\right]}{2^{n-D}\Gamma[\frac{n}{2}]}\frac{1}{|\boldsymbol{\ell}|^{D-n}} , \qquad \text{for } n \geq 3$$
$$J^{ij}_{(2)|\boldsymbol{\ell}} = (GM)^2\pi^{\frac{D}{2}}2^{D-3}\left[-\frac{2\Gamma[\frac{D}{2}]\ell^i\ell^j}{|\boldsymbol{\ell}|^D} + \frac{\Gamma\left[\frac{D-2}{2}\right]\delta^{ij}}{|\boldsymbol{\ell}|^{D-2}}\right] .$$

- One can show the followings by using the **induction**

Arbitrary rank $n \geq 5$ — J^{00}

- We can show that the off-shell currents at an arbitrary order n by induction.
- The corresponding recursion: $J^{00}_{(n)|\boldsymbol{\ell}} = \mathscr{E}^{[1]}_{(n)|\boldsymbol{\ell}} - \mathscr{E}^{[2]}_{(n)|\boldsymbol{\ell}}$,

even

$$\mathscr{E}^{[1]}_{(2n)|-\boldsymbol{\ell}_1} = (GM)^{2n}\frac{\ell_1^i}{|\boldsymbol{\ell}_1|^2}\left(-X^i_{(2n)|-\boldsymbol{\ell}_1} + Y^i_{(2n)|-\boldsymbol{\ell}_1}\right),$$

$$\mathscr{E}^{[2]}_{(2n)|-\boldsymbol{\ell}_1} = (GM)^{2n}\frac{\ell_1^i}{|\boldsymbol{\ell}_1|^2}\int_{\ell_2}\left(-X^j_{(2n-2)|-\boldsymbol{\ell}_{12}} + Y^j_{(2n-2)|-\boldsymbol{\ell}_{12}} - \ell^j_{12}J^{00}_{(2n-2)|-\boldsymbol{\ell}_{12}}\right)J^{ij}_{(2)|\boldsymbol{\ell}_2}.$$

odd

$$\mathscr{E}^{[1]}_{(2n+1)|-\boldsymbol{\ell}_1} = -(GM)^{2n+1}\frac{\ell_1^i}{|\boldsymbol{\ell}_1|^2}X^i_{(2n+1)|-\boldsymbol{\ell}_1},$$

$$\mathscr{E}^{[2]}_{(2n+1)|-\boldsymbol{\ell}_1} = (GM)^{2n+1}\frac{\ell_1^i}{|\boldsymbol{\ell}_1|^2}\int_{\ell_2}\left(-X^j_{(2n-1)|-\boldsymbol{\ell}_{12}} - \ell^j_{12}J^{00}_{(2n-1)|-\boldsymbol{\ell}_{12}}\right)J^{ij}_{(2)|\boldsymbol{\ell}_2}.$$

- Performing the bubble integrals and substituting, we have

$$J^{00}_{(2n)} = \frac{8(GM)^{2n}\pi^{\frac{D}{2}}2^{D-2n}\Gamma[\frac{D}{2}-n]}{\Gamma[n]}\frac{1}{|\boldsymbol{\ell}|^{D-2n}},$$

$$J^{00}_{(2n+1)} = \frac{8(GM)^{2n}\pi^{\frac{D}{2}}2^{D-2n-1}\Gamma[\frac{D-2n-1}{2}]}{\Gamma[n+\frac{1}{2}]}\frac{1}{|\boldsymbol{\ell}|^{D-2n-1}},$$

Arbitrary rank $n \geq 5$ — J^{ij}

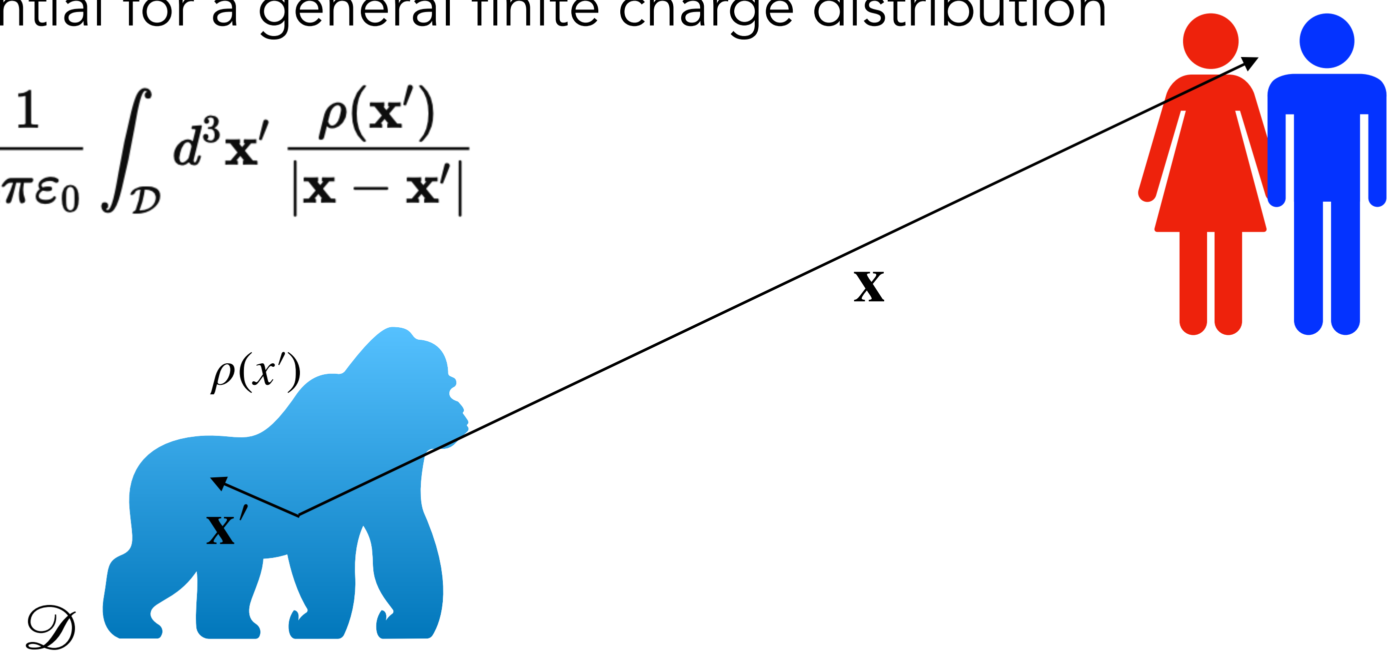
- The EoM for the spatial components

$$\Delta h^{ij}_{(n)} = \sum_{m=1}^{n-1} 4d^{i}_{(n-m)} d^{j}_{(m)} + 2\sigma^{ij}\partial_k d^k + Z_{(n)}^{k(i}{}_{k}{}^{j)} - 2Z_{(n)}^{(i|k|}{}_{k}{}^{j)} + \frac{1}{2}Z_{(n)}^{(i|k|j)}{}_{k} + \frac{1}{2}W_{(n)}^{(ij)}$$
$$- \left(Z^{k(i}_{(n-2)kl} - 2Z^{(i|k|}_{(n-2)kl} + \frac{1}{2}Z^{(i|k|}_{(n-2)lk} + \frac{1}{2}W^{(i|l|}_{(n-2)} \right) h^{j)l}_{(2)}$$

- Divide the EoM into **3 sectors**: d-sector, W-sector and Z-sector
- Interestingly, these three sectors vanish individually (**induction**).
- This implies $J^{ij}_{(n)|\ell} = 0$, as we expected
- This shows the all-order perturbative expansion satisfies the Einstein equation.

Multipole expansion

- Hopeless to find the exact potential for a general finite charge distribution
- Electrostatic potential: $\Phi(\mathbf{x}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_{\mathcal{D}} d^3\mathbf{x}' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$



- Spherical harmonics: $\dfrac{1}{|\mathbf{x} - \mathbf{x}'|} = \displaystyle\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} Y^*_{\ell m}(\hat{\mathbf{n}}') \, Y_{\ell m}(\hat{\mathbf{n}}), \quad r' < r,$
- multipole expansion: $\Phi(\mathbf{x}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+1}} \, Q_{\ell m} \, Y_{\ell m}(\hat{\mathbf{n}}),$

$$Q_{\ell m} \equiv \int d^3\mathbf{x}' \, r'^{\ell} \, Y^*_{\ell m}(\hat{\mathbf{n}}') \, \rho(\mathbf{x}').$$

- For the perturbative GR computation, it is convenient to use the Cartesian coordinates (loop integral in spherical coordinates?!)

STF tensor representation

- Consider the following rank-ℓ symmetric trace-free (STF) tensor with a unit vector $n^i = x^i/r$

$$\hat{n}^L := n^{\langle i_1} n^{i_2} \ldots n^{i_\ell \rangle}$$

- These are in one-to-one correspondence with the spherical harmonics

$$Y_{\ell m}\left(\theta_1, \phi_1\right) = Y^L_{\ell m} \hat{n}^L \left(\theta_1, \phi_1\right) \, , \quad \hat{n}_L = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y^L_{\ell m} Y_{\ell m}$$

where $Y^L_{\ell m} = Y^{i_1 i_2 \cdots i_\ell}_{\ell m}$ are constants.

- Multipole moment in STF basis

$$Q_L \equiv \int d^3\mathbf{x}\, \rho(\mathbf{x})\, x^{\langle i_1} \cdots x^{i_\ell \rangle} = \int d^3\mathbf{x}\, \rho(\mathbf{x})\, r^\ell\, \hat{n}_L$$

GR case

- **Mass multipole moments**

$$M^L(t) = (-1)^\ell \int d^3x\, T^{00}(t,\mathbf{x})\, \hat{x}^L$$

- **Current multipole moments**

$$S^L(t) = (-1)^\ell \int d^3x\, \hat{x}_{L-1}\, \epsilon_{ijk} x^j T^{0k}$$
$$= (-1)^\ell \int d^3x\, \epsilon^{jk\langle i_l} x^{i_1} \cdots x^{i_{l-1}\rangle} x^j T^{0k}(t,\mathbf{x})$$

- For the stationary case, $T^{ij}_{\text{stationary}} = 0$

From the conservation, $\partial_i T^{ij} = 0$ and we have $\int \mathrm{d}^3x\, x^L \partial_i T^{ij} = 0,$

After integration by part,

$$0 = \int d^3x \partial_i \left(x^L T^{ij}\right) - \int d^3x T^{ij} \partial_i x^L$$
$$= \underbrace{\oint_{S_\infty} dS_i x^L T^{ij}}_{=0} - \int d^3x T^{ij} \underbrace{\partial_i x^L}_{x^{L-1}} \qquad \Longrightarrow T^{ij} = 0$$

MPM expansion

- **Post-Minkowskian expansion**: power expansion with respect to Newton's constant G

$$\mathfrak{g}^{\mu\nu}(x^\mu) \equiv \sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} - \sum_{n=1}^{\infty} G^n h^{\mu\nu}_{(n)},$$

- Multipole expansion for the n-PM field $h^{\mu\nu}_{(n)}$

$$h^{\mu\nu}_{(n)}(x^\mu) = \sum_{l=0}^{l_{\max}} h^{\mu\nu}_{(n)L}(r,t)\, \hat{n}^L$$

- In the Multipolar post-Minkowskian (MPM) framework, $h^{\mu\nu}_{(n)}$ is expanded in terms of the mass and current multipoles

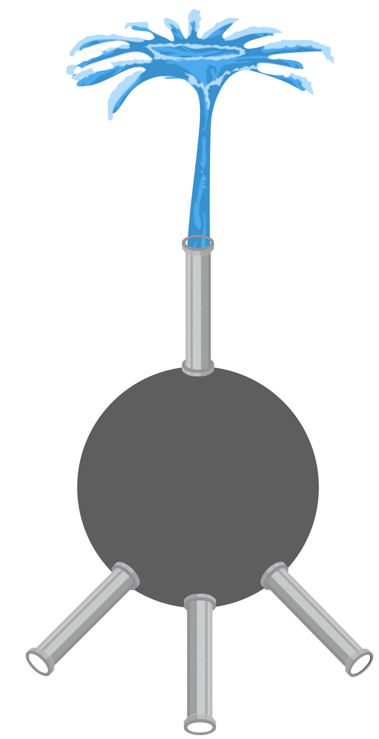
$$\{M_L, S_L\}, \quad L = i_1 \cdots i_\ell.$$

Summary and future directions

- **Established a new computational framework for perturbative GR**
  - defined a “good” variable — tensor density & doubled metric
  - derived a recursion relation in a remarkably simple form — no infinite expansion
  - Showed the integral factorization occurs — only bubble integrals arise
  - Derived the Schwarzschild BH solution all orders in Newton constant
- **Applications**
  - Extension to binary black holes — two moving point masses
  - Kerr BH and stars (gravitational multipole expansion)
  - Finding interesting unknown solutions — physically intuitive setup.
  - Computing scattering amplitudes for QCD/SM

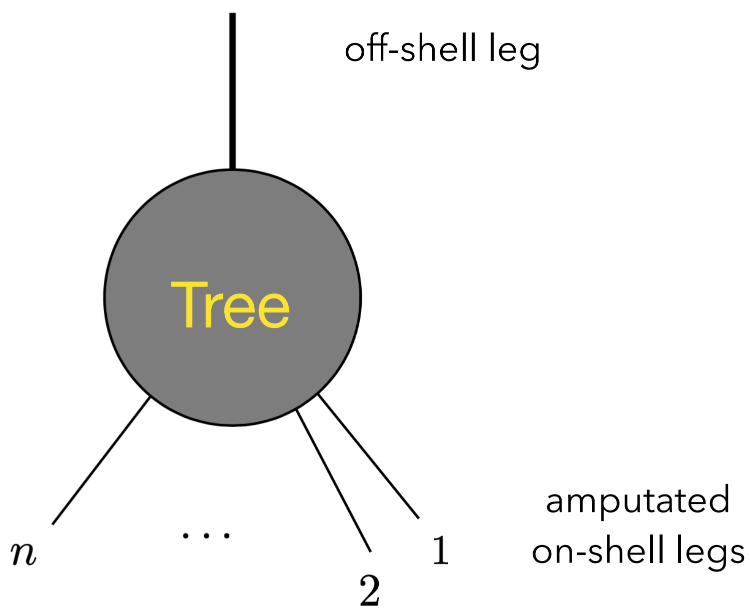
Thank you!

Off-shell Currents



- **Off-Shell recursions**: recursions for **off-shell currents** [Berends, Giele '87] for gluon amplitude at **tree-level**
- **Rank-n Off-shell currents: sum of all $(n+1)$-point Feynman diagrams**
- Diagrammatic representation

$$\Phi_{12\cdots n} =$$

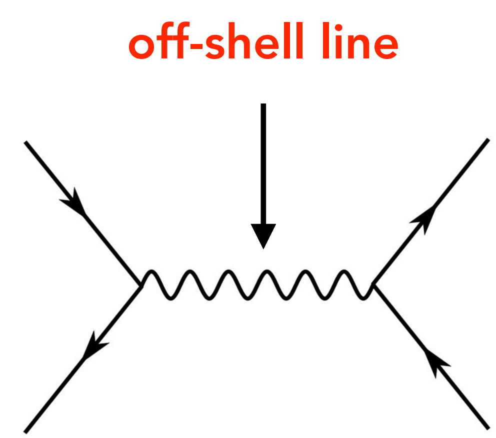


The off-shell line satisfy the conservation law

$\partial_\mu J^\mu_{12\cdots n} = 0$ without EoM — **Ward identity**

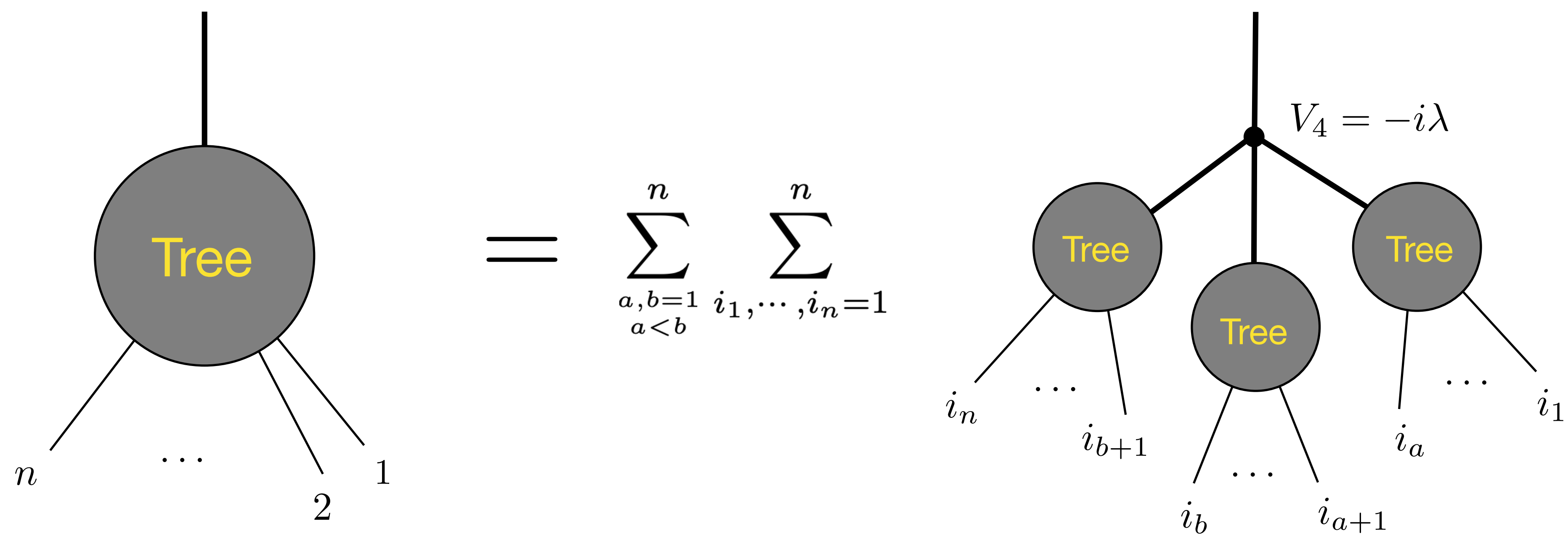
- Off-shell lines can be **glued** in a specific way (interaction vertices)

  Intermediate states are off-shell



Off-shell Recursion [Berends, Giele '87]

- **Recursions**: **hidden self-similarity** — finite number of interaction vertices (**patterns**)
- Identifying the **Hierarchy** for off-shell currents: **# of on-shell legs**
- ϕ^4 theory:



- **Efficiency**:
  - Do not treat individual diagrams
  - **Recycling** calculations - never repeat the same calculations!

- **Gravity** — **infinite number of vertices** (No patterns)

Field equations - temporal component

- Under the **static condition** and **harmonic gauge**, the $\mathscr{G}^{00}$ component yields

$$\mathscr{G}^{00}=\frac{1}{4}\sigma^{00}\partial_i\left[\sigma^{ij}\left(\tilde{\sigma}_{00}\partial_j\sigma^{00}-\tilde{\sigma}_{kl}\partial_j\sigma^{kl}\right)\right]=\frac{1}{2}j^{00}$$

- The source term is only relevant to the 1st order: $\Box h^{00}=-2j^{00}$

  It is sufficient to solve the **simplified form** for higher-order,

$$\partial_i\left[\sigma^{ij}\left(\tilde{\sigma}_{00}\partial_j\sigma^{00}-\tilde{\sigma}_{kl}\partial_j\sigma^{kl}\right)\right]=0$$

- Substituting the expansion of the fields, we derive the field equation for h^{00}

Laplacian $\longrightarrow$ $\Delta h^{00}_{(3)}=\partial_i\left(X^i_{(3)}+h^{ij}_{(2)}\partial_j h^{00}_{(1)}\right),$

$$\Delta h^{00}_{(4)}=\partial_i\left(X^i_{(4)}-Y^i_{(4)}-X^j_{(2)}h^{ij}_{(2)}+\partial_j h^{00}_{(2)}h^{ij}_{(2)}+\partial_j h^{kk}_{(2)}h^{ij}_{(2)}\right),$$

$n\geq 5$, Fixed form! $\quad \Delta h^{00}_{(n)}=\partial_i\left(X^i_{(n)}-Y^i_{(n)}\right)-\partial_i\left(X^j_{(n-2)}-Y^j_{(n-2)}-\partial_j h^{00}_{(n-2)}\right)h^{ij}_{(2)}$

where $\quad X^i_{(n)}=\sum_{m=1}^{n-1}\tilde{h}^{00}_{(n-m)}\partial^i h^{00}_{(m)}\quad$ and $\quad Y^i_{(n)}=\tilde{h}^{kl}_{(n-2)}\partial^i h^{kl}_{(2)}$

Field equations - spatial components

- **Spatial components for Einstein eq**

$$\mathscr{G}^{ij} = \frac{1}{2}\sigma^{kl}\partial_k\partial_l\sigma^{ij} - \sigma^{k(i}\partial_k\partial_l\sigma^{j)l} + \frac{1}{4}\left(2Z^{k(i}{}_{kl} - 4Z^{(i|k|}{}_{kl} + Z^{(i|k|}{}_{lk} + W^{(i}{}_{l}\right)\sigma^{j)l}$$
$$+(D-2)d^i d^j + \frac{1}{2}\partial_k\sigma^{kl}\partial_l\sigma^{ij} - \frac{1}{2}\partial_l\sigma^{ki}\partial_k\sigma^{lj} + \sigma^{ij}\partial_k d^k = 0$$

where $Z^{ij}{}_{kl} = \sigma^{im}\partial_m\sigma^{jn}\partial_k\tilde{\sigma}_{nl}\,, \qquad W^i{}_j = \sigma^{ik}\partial_k\sigma^{00}\partial_j\tilde{\sigma}_{00}\,, \qquad d^i = \sigma^{ij}\partial_j\hat{d}\,.$

- No source in spatial components, $j^{ij} = 0 \Longrightarrow$ EoM: $\mathscr{G}^{ij} = 0$

- Substituting the expansion of the fields, we derive the **field equation for h^{ij}**

$$\Delta h^{ij}_{(3)} = 8d^{(i}_{(2)}d^{j)}_{(1)} + 2\delta^{ij}\partial_k d^k_{(3)} - 2h^{ij}_{(2)}\partial_k d^k_{(1)} + \frac{1}{2}W^{ij}_{(3)} - \frac{1}{2}h^{k(i}_{(2)}W^{j)k}_{(1)}\,,$$

$$\Delta h^{ij}_{(4)} = h^{kl}_{(2)}\partial_k\partial_l h^{ij}_{(2)} - 2h^{k(i}_{(2)}\partial_k\partial_l h^{j)l}_{(2)} + \frac{1}{2}\left(2Z^{k(i}_{(4)}{}_{k}{}^{j)} - 4Z^{(i|k|}_{(4)}{}_{k}{}^{j)} + Z^{(i|k|j)}_{(4)}{}_{k} + W^{(ij)}_{(4)}\right) + \cdots\,,$$

$n \geq 5$
Fixed form!

$$\Delta h^{ij}_{(n)} = \sum_{m=1}^{n-1} 4d^i_{(n-m)}d^j_{(m)} + 2\sigma^{ij}\partial_k d^k_{(n)} + Z^{k(i}_{(n)}{}_{k}{}^{j)} - 2Z^{(i|k|}_{(n)}{}_{k}{}^{j)} + \frac{1}{2}Z^{(i|k|j)}_{(n)}{}_{k} + \frac{1}{2}W^{(ij)}_{(n)} + \cdots$$